CMPE 598 - Quantum Algorithms
Lecture Notes - 13 February 2018

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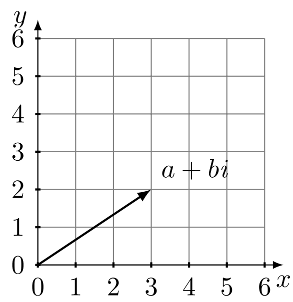
The observation probability of any state is the square of the modulus (| . |) of its amplitude.



$$| a + bi | = \sqrt{a^2 + b^2}$$

- Physics ensure that modulus values add up to 1.
- Every Quantum algorithm can be implemented only by using Real Numbers.

  In some cases using imaginary numbers makes the task easier but does not increase the capabilities of Quantum computers.
- Until now only small Quantum computers have been successfully implemented.
- Some say it will never be possible to build Quantum Computers because of noise. Proving unimplementability of them would be a major discovery as well!!

Probabilistic machines are best represented by stochastic matrices which are square matrices of size nxn where n is the # of states.

$$\begin{pmatrix} . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & 0.72 & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{pmatrix} \quad P(S_4 \rightarrow S_3) = 0.72$$

Strictest Possible Requirements

Definition 0.1. rt-QFA A real-time Quantum Finite Automaton(rtQFA) is a 5-tuple $(Q, \Sigma, q_1, \{E_\sigma\}_{\sigma\in\Sigma}, F)$ where Q is the set of states, Σ is the input alphabet, q_1 is the initial state, $\{E_\sigma\}_{\sigma\in\Sigma}$ is the transition matrices and F is the set of accept states.

Each $\{E_\sigma\}$ will be a collection of $m \mid Q \mid * \mid Q \mid$ dimensional matrices called "operational elements " for some $m \geq 0$. So for each letter we have m matrices which corresponds to the maximum number of probabilistic branches.

$$\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \overleftarrow{\frac{1}{2}} \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \overrightarrow{\frac{1}{2}} \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad OR \quad \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \rightarrow \begin{pmatrix}0\\\frac{1}{2}\\\frac{1}{2}\\0\end{pmatrix}$$

The second representation allows to represent the tree in a single vector.

So because of m we have two different stochastic processes. (1) The stochastic transition matrix and (2) which transition matrix to use.

Simple QFA Example

- State set : $\{q_1, q_2, q_3\}$
- $\Sigma = \{a\}$
- $m = 2$

$$E_{a,1} = \begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & 0\\\frac{1}{\sqrt{2}} & 0 & 0\\0 & \frac{1}{\sqrt{2}} & 0\end{pmatrix} \quad E_{a,2} = \begin{pmatrix}0 & \frac{1}{\sqrt{2}} & 0\\0 & 0 & 0\\0 & 0 & 1\end{pmatrix}$$

At the end probabilities add up to 1 according to some criterion.

Step 1: initially $\begin{pmatrix}1\\0\\0\end{pmatrix}$ multiply with matrices $E_{a,1} * \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{pmatrix}$ and $E_{a,2} * \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

In tree representation:

$$\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{pmatrix} \overleftarrow{\text{"1"}} \begin{pmatrix}1\\0\\0\end{pmatrix} \overrightarrow{\text{"2"}} \begin{pmatrix}0\\0\\0\end{pmatrix}$$

So classical probabilities are included in the values in the matrix.

-We can only impose unitarity to closed systems such as our universe.
-The room the Quantum computer sits in is unitary not the computer itself.

Step 2:

$$E_{a,1}* \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \quad P_1 = \frac{3}{4} \quad and \quad E_{a,2} * \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \quad P_2 = \frac{1}{4}$$

So it is like a biased coin branching:

$$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \overleftarrow{\frac{3}{4}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \overrightarrow{\frac{1}{4}} \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}$$

So we end up with the above situation after two iterations without observing.

Since the probabilities do not add up to 1 we normalize.

Procedure for normalizing:

Divide the vector by the square root of the branching probability.

$$"1" \quad \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \xrightarrow{\sqrt{\frac{3}{4}}} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \quad "2" \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \xrightarrow{\sqrt{\frac{1}{4}}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

<u>Question</u>: Probability to accept after input "aa"?

probability of being in q_2:

$$\frac{3}{4}.(\frac{1}{\sqrt{3}})^2 = \frac{1}{4}$$

If you consider the transition matrices as a big rectangular matrix, probabilities in each column add up to 1 and all columns are orthogonal.

$$E_{a,1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$E_{a,2} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

-All columns have length 1 and they are pairwise orthogonal.
-The unitary property can be achieved by adding extra columns which represent the rest of the closed system.

Density Matrix

We have 2 types of ignorances : Classical probabilistic ignorance and Quantum ignorance. We can use density matrix to represent both in a single representation.
$|\psi>$: represents a column vector
$<\psi|$: represents row version with conjugates of each entry is replaced.

$$|\psi> = \begin{pmatrix} \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \\ . \\ . \\ . \end{pmatrix} \quad \rightarrow \quad <\psi| = \begin{pmatrix} \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} & . & . & . \end{pmatrix}$$

The density matrix representation : $\{(p_i, |\psi_i>) | 1 \leq i \leq n\}$

$$P = \sum_i p_i |\psi_i><\psi_i|$$

Then for the example above we have n=2:

$$(1) i = 1 \quad \frac{3}{4} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \quad (2) i = 2 \quad \frac{1}{4} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$So \quad P = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

-Diagonal entries are all real numbers between $0, 1$ and add up to 1, they represent transition probabilities.

-Calculating the density matrix of the next state if you are at P and read a:

$$P' = \sum_i E_{a,i} * P * E_{a,i}^t$$

where $E_{a,i}^t$ is the complex conjugate of $E_{a,i}$, the transition matrix.

So again by using the first example, starting from the initial state:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- In order to further iterate just change the P inside with the newly found version.
- This is the fundamental syntax of QFAs, from this we can go to more advanced versions.
- So all succeeding programs must satisfy,

  (1) probability add up to 1.

  (2) columns should be orthogonal.

Question: Can we convert a function represented by a PFA with this model?

PFA has stochastic transition matrices $(Q, \Sigma, q_1, \Sigma_{\sigma \in \Sigma}, F)$,

$$T = \begin{pmatrix} 0.1 & 0.7 & 0.2 \\ 0.5 & 0.3 & 0.2 \\ 0.4 & 0 & 0.6 \end{pmatrix}$$

how to convert it into a Quantum Matrix?

Create 3 matrices,

$$E_{a,1} = \begin{pmatrix} \sqrt{0.1} & 0 & 0 \\ \sqrt{0.5} & 0 & 0 \\ \sqrt{0.4} & 0 & 0 \end{pmatrix}$$

$$E_{a,2} = \begin{pmatrix} 0 & \sqrt{0.1} & 0 & \\ 0 & \sqrt{0.5} & 0 & 0 \\ 0 & 0 & 0 & \end{pmatrix}$$

$$E_{a,3} = \begin{pmatrix} 0 & 0 & \sqrt{0.2} \\ 0 & 0 & \sqrt{0.2} \\ 0 & 0 & \sqrt{0.6} \end{pmatrix}$$

If we call quantum version of T as T^Q then,

$$T^Q \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x\sqrt{0.1} \\ x\sqrt{0.5} \\ x\sqrt{0.4} \end{pmatrix}, \quad \begin{pmatrix} y\sqrt{0.7} \\ y\sqrt{0.3} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} z\sqrt{0.2} \\ z\sqrt{0.2} \\ z\sqrt{0.6} \end{pmatrix}$$

remember that $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ corresponds to $\begin{pmatrix} x^2 \\ y^2 \\ z^2 \end{pmatrix}$ in probability values.

Being in the state q_1 is $0.1x^2 + 0.7y^2 + 0.2z^2$ which satisfies the PFA results. So QFAs can recognize all regular languages.

Note: If transitions are unitary we can reverse the process. DFAs lack this property, we may have multiple incoming arrows with the same letter to a single state.

Next Class: Quantum Supremacy! Express non-regular languages or express same language in less states.