CMPE 598 - Lecture Notes

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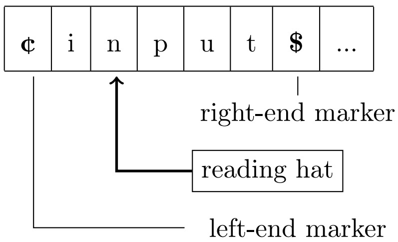
1 Two-Way Finite Automata

A *two-way deterministic finite automaton* (**2DFA**) is a generalized DFA which can read the input string in two directions. It has a *reading hat* which can move one character left or right over the input string. The string has two delimiters ¢ and \$, which are not elements of the input alphabet Σ, and respectively showing the beginning and the end of the string.



Transition functions: $Q \times \Sigma \rightarrow Q \times \{L, R\}$
System reads a character in a state, then decides to go left or right and switches its state.
State set: There are *halting* states (accept or reject) in which the reading hat reaches to the right-end and automaton stops, and *non-halting* states where it does not.

Can it recognize non-regular languages?

No, 2DFAs and DFAs are equivalent in the sense of only recognizing regular languages. But they might have some practicalities over DFAs.
Example: Consider the language $L_m = \{a^i \mid \text{i is a multiple of } m\}$ and take $m = 2 \cdot 3 \cdot 5 \cdot ... \cdot 19$. You can build a 2DFA which traverses the string and for one prime factor of m such as 5, it counts the string length in modulo 5 and checks its divisibility by using only 5 states. By doing this for every factor of m, L_m can be recognized with only 2 + 3 .. + 19 states, whereas a DFA needs at least m number of states.

A proof to their equivalence: Let M be a 2DFA with set of states S recognizing L.
For every string $w \in \Sigma^*$, define a function $\tau_w \colon \{\overline{s}_0\} \cup S \rightarrow \{0\} \cup S$.
This function will serve the purpose of showing how the machine behaves when it crosses the boundary between w and z in an input like | ¢ | w | z | \$ |

For any state $s \in S$, $\tau_w(s)$ shows the ultimate result of the motion of M when it starts on the right-most symbol of w in state s, i.e. if M ultimately leaves w after some steps, from the right-most symbol into the state s', then $\tau_w(s) = s'$.
If M never leaves w or leaves it from the left, then $\tau_w(s) = 0$.

$\tau_w(\overline{s}_0)$ has the special meaning of which state the machine lands in when it leaves w for the first time from the right-most symbol i.e. when M is started on the initial state on the left most symbol of w, eventually it leaves w from the right-most symbol into the state s' where $\tau_w(\overline{s}_0) = s'$. or it never leaves w where $\tau_w(\overline{s}_0) = 0$.

Now consider two input strings $w_1 z$ and $w_2 z$. If M accepts a string, the marker ends up at right-end symbol \$, so it leaves w from its right-most symbol at least once and lands in the left-most symbol of z. Let τ_{w_1} and τ_{w_2} denote the "table"s of τ meaning all values of τ for any element from $\{\overline{s}_0\} \cup S$. If $\tau_{w_1} = \tau_{w_2}$ for these two strings w_1 and w_2, then they land in the left-most symbol of z and into the same state, as their tables are the same. So, M either accepts or rejects both, which means for all $z \in L$, $w_1 z \equiv_L w_2 z \iff w_1 \equiv_L w_2$.

Note also that, if there are k states in S, there can be only $(k+1)^{k+1}$ distinct tables, so there are only a finite number of different τ functions!

Using Myhill-Nerode theorem $\implies$ Only regular languages are recognized by 2DFAs.

2 Two-way Probabilistic Finite Automata

Let us build a 2PFA which recognizes $\{a^n b^n \mid n \geq 0\}$ with error bound ε for any desired $\varepsilon > 0$.

Let $x = \frac{\varepsilon^2}{2}$. The machine splits into three paths when it starts.
All paths check whether an "a" appears after a "b" while doing their other jobs, and reject if this happens.

Path 1 moves on with probability x and restarts with probability $1 - x$ (i.e. goes back to ¢ and switches to the initial state) when reading symbols a or b.
After reading the right-end marker, it accepts with probability 1.

Path 2 moves on with probability x^2, restarts with probability $1 - x^2$ when reading symbol a.
On b's, it goes on with probability 1.
At the right-end marker, it rejects with probability $\frac{\varepsilon}{2}$ and restarts with $1 - \frac{\varepsilon}{2}$.

Path 3 is just like **Path 2**, but transitions between a and b are interchanged.

If the input is of the form $a^m b^n$, then the accept and reject probabilites in the first round -before any restart- are as follows:

$$P_{acc} = \frac{1}{3} x^m x^n \qquad\qquad P_{rej} = \frac{1}{3} \cdot \frac{\varepsilon}{2} x^{2m} + \frac{1}{3} \cdot \frac{\varepsilon}{2} x^{2n} = \frac{\varepsilon}{6}(x^{2m} + x^{2n})$$

If $m = n$,

$$\frac{P_{rej}}{P_{acc}} = \frac{\frac{\varepsilon}{6}(x^{2m} + x^{2m})}{\frac{1}{3} x^{2m}} = \varepsilon$$

.
If $m \neq n$, without loss of generality, $m = n + d$ for $d > 0$. Then,

$$\frac{P_{acc}}{P_{rej}} = \frac{2x^{2n+d}}{\varepsilon \cdot (x^{2n+2d} + x^{2n})} = \frac{2}{\varepsilon} \cdot \frac{x^d}{x^{2d} + 1} < \frac{2}{\varepsilon} x^d \leq \frac{2}{\varepsilon} x$$

Substitute $x = \frac{\varepsilon^2}{2}$:

$$\implies \frac{P_{acc}}{P_{rej}} < \varepsilon$$

So this 2PFA can recognize this non-regular language with an error bound ε, but there is a catch: its runtime is bad *(Frey)*. For any polynomial p, 2PFA with expected runtime $\theta(p(n))$ recognize only the regular languages with bounded error!