CMPE 300 - Analysis of Algorithms
Fall 2016
Assignment 1

Solutions

Question 1

(40)

For this problem, you should use the definition $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$.

1. Show that $e^x \geq 1 + x$ for all real $x \geq 0$ and also that $e^k \geq \frac{k^k}{k!}$ for $k \in \mathbb{Z}^+$.
   Solution:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \\ &\geq 1 + x. \end{aligned}$$

   since $x \geq 0$ and all the terms in the expansion are non-negative.

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{k^k}{k!} + \ldots \\ &\geq \frac{k^k}{k!}. \end{aligned}$$

   since $k \in \mathbb{Z}^+$ and the term $\frac{k^k}{k!}$ must appear in the expansion when x is equal to k.

2. Prove that $\binom{n}{k} \geq \left(\frac{n}{k}\right)^k$ for integers $0 < k < n$.
   Solution:

$$
\begin{aligned}
\binom{n}{k} &= \frac{n!}{(n-k)!k!} \\
&= \frac{n(n-1)(n-2)\dots(n-k+1)}{k(k-1)(k-2)\dots 1} \\
&= \left(\frac{n}{k}\right)\left(\frac{n-1}{k-1}\right)\dots\left(\frac{n-k+1}{1}\right) \\
&\geq \left(\frac{n}{k}\right)\left(\frac{n}{k}\right)\dots\left(\frac{n}{k}\right) \qquad \text{since } \tfrac{n-i}{k-i} \geq \tfrac{n}{k} \text{ for all } 0 \leq i \leq k-1 \\
&= \left(\frac{n}{k}\right)^k.
\end{aligned}
$$

We have proved that $\binom{n}{k} \geq \left(\frac{n}{k}\right)^k$ for integers $0 < k < n$.

3. Prove that $\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$ for integers $0 < k < n$.
   Solution:

$$
\begin{aligned}
\binom{n}{k} &= \frac{n!}{(n-k)!k!} \\
&= \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} \\
&\underline{\leq n \cdot n \cdot n \dots n} \\
&\quad\; k! \\
&= \frac{n^k}{k!} \\
&= \frac{n^k}{k!} \cdot \frac{k^k}{k^k} = \frac{n^k}{k^k} \cdot \frac{k^k}{k!} \\
&\leq \frac{n^k}{k^k} \cdot e^k \qquad \text{by part a)} \\
&= \left(\frac{en}{k}\right)^k.
\end{aligned}
$$

We have proved that $\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$ for integers $0 < k < n$.

4. Prove that $k \ln \frac{n}{k} \leq \ln \binom{n}{k} \leq k \ln \frac{n}{k} + k$ for integers $0 < k < n$.
   Solution: Combining what we have found in part b) and c), we obtain

$$
\begin{aligned}
\left(\frac{n}{k}\right)^k \leq \quad & \binom{n}{k} \leq \left(\frac{en}{k}\right)^k \\
k \ln \frac{n}{k} \leq \ln & \binom{n}{k} \leq k \ln \frac{n}{k} + k
\end{aligned}
$$

for all integers $0 < k < n$.

Question 2

(60) Do a fine (exact) analysis and calculate the number of operations for the worst case and average case. Express the worst case and average case time complexities of the function using the big O notation. The basic operations are assignments and condition checks, which each consume 1 unit of time. You can ignore the assignments in the loop headers.

```
Function: (X[0 : n − 1],k)
Input:  X[0 : n − 1] array of elements where X[i] ∈ {0, 1},
        k a nonnegative integer less than or equal to n
Output: A number
  for i ← 0 to n − 1 do
    for j ← 0 to i do
      a ← a + X[j]
    end for
    A[i] ← a/(i + 1)
  end for
  for i ← 0 to n − 1 do
    b ← b + A[i]
  end for
  if b = k then
    for each subset S of {0, 1, . . . , n − 1} do
      for j ← k^k do
        d ← d + 1
      end for
    end for
    return d
  end if
  else
    return b
  end if
end
```

Let us analyze the first for loop.

$$\sum_{i=0}^{n-1}\Big(\sum_{j=0}^{i} 1 + 1\Big) = \sum_{i=0}^{n-1}(i+1) + 1 = \sum_{i=0}^{n-1}(i+2)$$
$$= \sum_{i=2}^{n+1} i$$
$$= \sum_{i=1}^{n+1} i - 1$$
$$= \frac{(n+1)(n+2)}{2} - 1$$
$$= \frac{n^2+3n}{2}$$

Let us analyze the second for loop.

$$\sum_{i=0}^{n-1} 1 = n$$

Let us analyze the for loop inside the if statement.

$$\sum_{i=1}^{2^n}\left(\sum_{j=1}^{k^k} 1\right) = \sum_{i=1}^{2^n}(k^k)$$
$$= 2^n k^k$$

In the worst case the first two loops and the loop inside the statement is executed. Total number of operations and the big-O complexity is given by:

$$\frac{n^2+3n}{2} + n + 2^n k^k = \frac{n^2+5n}{2} + 2^n k^k \in O(2^n k^k)$$

Note that here k is not a constant, but an input integer less than or equal to n. You may also replace k with n since $k \leq n$.

For the average case, let us calculate the probability p that the if condition is executed. If condition is executed when b is equal to k, that is when the sum of the numbers in the input array X adds up to k. Since $X[i] \in \{0,1\}$, this probability is equal to probability of k successes in n trials which has binomial distribution. Therefore,

$$p = \binom{n}{k}\frac{1}{2^n}.$$

Let us count the total number of operations for the average case. Note that the number of operations is 0 when the if condition is not executed.

$$
\begin{aligned}
\frac{n^2+5n}{2} + p(2^n k^k) &= \frac{n^2+5n}{2} + \binom{n}{k}\frac{1}{2^n}(2^n k^k) \\
&= \frac{n^2+5n}{2} + \binom{n}{k}k^k
\end{aligned}
$$

We can use the inequality in Question 1 to evaluate $\binom{n}{k}$.

$$
\begin{aligned}
\frac{n^2+5n}{2} + \binom{n}{k}k^k &\leq \frac{n^2+5n}{2} + \left(\frac{en}{k}\right)^k \\
&= \frac{n^2+5n}{2} + (en)^k \in O(n^k) \quad \text{(as } n \text{ and } k \to \infty\text{)}
\end{aligned}
$$